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Resonant oscillations in shallow water with small mean-square disturbances

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The ordinary differential equation $\epsilon^2\ddot{y}+\epsilon^2\delta\dot{y} = y^2-\cos t-1-c$, which represents forced water waves on shallow water near resonance is considered when the dispersion ϵ and the constant c are small. Asymptotic and numerical methods are used to show that solutions which are bounded for all finite t can exist only if $c > -2^{-\frac{2}{3}}\epsilon^{\frac{4}{3}}\times 1.4664\ldots$.

1. Introduction

The study of the problem of resonant sloshing of water in a horizontally oscillated container of finite depth was initiated by Chester (1968). He derived the integral equation

$$\alpha_1+\cos t = (f(t)-\alpha_2)^2-\alpha_3\int_{-\infty}^{\infty}(f(t-\tau)-f(t))\ln\tanh(\alpha_4|\tau|)\,\mathrm{d}\tau-\alpha_5\int_0^{\infty}f(t-\tau)\,\tau^{-\frac{1}{2}}\,\mathrm{d}\tau, \tag{1.1}$$

for the periodic response near one of the resonant frequences. For our purpose, it is sufficient to say that the α_i are constants and the function $f(t)$ has period 2π and zero mean. For further details see Chester (1968). In the case of shallow water and in the absence of damping Ockendon & Ockendon (1973), Cox & Mortell (1983) and Miles (1985) have shown that this reduces to a second-order differential equation:

$$\tfrac{1}{3}k^2(\ddot{f}+f)-\lambda f-\tfrac{3}{2}f^2+\frac{2}{\pi}\cos t = \text{constant}. \tag{1.2}$$

Ockendon, Ockendon & Johnson (1986) also consider a more tractable model for damping and introduce a term proportional to $\dot{f}$ to the left-hand side of (1.2). The condition that f has zero mean over the interval $(-\pi, \pi)$ means that the constant on the right-hand side of (1.2) is equal to

$$-\frac{3}{4\pi}\int_{-\pi}^{\pi}f^2\,\mathrm{d}t.$$

In (1.2), k of order one represents the 'Korteweg–de Vries' scalings where the nonlinearity of the resonant response is of the same order of magnitude as the dispersive effects. Ockendon *et al.* (1986) considered the case of small k which occurs when the water depth h tends to zero. It is convenient in this case to write

$$f = -\frac{\lambda}{3}+\frac{k^2}{9}+\left(\frac{4}{3\pi}\right)^{\frac{1}{2}}y, \quad \hat{\lambda} = 3^{\frac{1}{2}}\pi^{\frac{3}{2}}\lambda, \tag{1.3}$$

so that, when damping is included,

$$\epsilon^2\ddot{y}+\epsilon^2\delta\dot{y} = y^2-\cos t-1-c, \tag{1.4}$$

where

$$\epsilon = 2^{-\frac{1}{2}}3^{-\frac{3}{4}}\pi^{\frac{1}{4}}k, \tag{1.5}$$

and δ is a constant.

Then the detuning parameter $\hat{\lambda}$ and the constant c are given by

$$\hat{\lambda} = \int_{-\pi}^{\pi} y\, \mathrm{d}t+2\pi\epsilon^2 \tag{1.6}$$

and

$$c = \frac{1}{2\pi}\int_{-\pi}^{\pi} y^2\, \mathrm{d}t-1. \tag{1.7}$$

Ockendon *et al.* (1986) show that there are multiple solutions for equation (1.4) and discuss the character of the solutions. They also obtain the form of the response diagrams $c(\hat{\lambda})$ for the case c not small. For fixed values of ϵ and δ there are a finite number of solutions. In the case of zero damping these are labelled $(n, 1)$ and $(n, 0)$. The notation (i, j) indicates the number of 'spikes' in the solution $y(t)$ corresponding to rapid oscillations on a timescale of order $(\epsilon^{-1}t)$, with i being the number of spikes at $t = 0$ and j the number at $t = \pi$. Ockendon *et al.* (1986) present numerical evidence that the branches labelled $(n, 0)$ and $(n, 1)$ in the response diagram are in fact connected, the spike at $t = \pi$ continuously diminishing in size as c passes through the minimum value $c = c_n$. No analysis of this is given, however. The principal aim of this paper is to analyse this behaviour by means of constructions of an asymptotic solution to (1.4) as $\epsilon\to 0$. It is shown that $c = O(\epsilon^{\frac{4}{3}})$ in order to obtain the transition from the $(n, 0)$ solution to the $(n, 1)$ solution.

In §2 we find the appropriate form of approximate solution in the form of a perturbation series. We show that two approximations are required, one valid throughout the interval $(-\pi, \pi)$, apart from a small region near the end points, and an inner solution valid for values of t near $\pm\pi$. This requires the solution of the differential equation

$$\ddot{x} = x^2-T^2-\gamma, \tag{1.8}$$

where T is a scaled time variable. This equation has been solved numerically by Byatt-Smith (1988). In the final section we discuss the result and compare the solutions with those of Ockendon *et al.* (1986).

2. The construction of a 2π-periodic solution

2.1. *The form of an approximate solution*

We wish to study the behaviour of the solutions of the equation

$$\epsilon^2\ddot{y}+\delta\epsilon^2\dot{y} = y^2-\cos t-1-c, \tag{2.1}$$

where $\delta \geqslant 0$, $0<\epsilon \ll 1$ and $|c|$ is small, that is $o(1)$ as $\epsilon\to 0$. In particular, we seek initial conditions $y(0)$ and $\dot{y}(0)$ which give rise to a periodic solution with $y(2\pi+t) = y(t)$.

With c small, a solution of the form $y = \pm(1+\cos t)^{\frac{1}{2}} = \pm\sqrt{2}|\cos\frac{1}{2}t|$ is clearly invalid near $t = \pm\pi$, where derivatives become large. If $\epsilon^2\ddot{y}$ and y^2 are to be of comparable order then this suggests that timescales of order $O(\epsilon^{-1})$ are going to be important. Hence we look for a solution of the form

$$y = Y(\tau, t), \tag{2.2}$$

where

$$\tau = f(t)/\epsilon. \tag{2.3}$$

Here $f(t)$ is at our disposal and the aim, following the ideas of Kusmak (1959) is to choose f so that the first approximation to Y satisfies an equation whose solution is periodic in τ with period independent of t.

Then

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \frac{f'}{\epsilon}\frac{\partial}{\partial \tau}, \tag{2.4}$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} = \frac{\partial^2}{\partial t^2} + \frac{1}{\epsilon}\left(f''\frac{\partial}{\partial \tau} + 2f'\frac{\partial^2}{\partial \tau\,\partial t}\right) + \frac{f'^2}{\epsilon^2}\frac{\partial^2}{\partial \tau^2}, \tag{2.5}$$

so that
$$f'^2 Y_{\tau\tau} + \epsilon\{f'' Y_\tau + 2f' Y_{\tau t} + \delta f' Y_\tau\} + \epsilon^2\{Y_{tt} + \delta Y_t\} = Y^2 - \cos t - 1 - c. \tag{2.6}$$

We now seek a series solution of (2.6) of the form

$$Y = \sum_0^\infty \epsilon^i Y_i. \tag{2.7}$$

The coefficients of the powers of ϵ then yield

$$\epsilon^0: \quad f'^2 Y_{0\tau\tau} = Y_0^2 - 1 - \cos t. \tag{2.8}$$

Using the criterion outlined above we choose

$$f' = \surd 2(1+\cos t)^{\frac{1}{4}} = 2^{\frac{3}{4}}|\cos \tfrac{1}{2}t|^{\frac{1}{2}}, \tag{2.9}$$

so that if
$$Y_0 = (1+\cos t)^{\frac{1}{2}} X_0, \tag{2.10}$$

X_0 will satisfy
$$2X_{0\tau\tau} = X_0^2 - 1, \tag{2.11}$$

or
$$X_{0\tau}^2 = \tfrac{1}{3}X_0^3 - X_0 + \text{constant}. \tag{2.12}$$

If (2.11) is regarded as an ordinary differential equation, the solutions $X_0(\tau, a, \phi)$ are periodic and have as constants of integration the amplitude a and phase ϕ. The period τ_0 of these solutions depends only on a and lies between 2π, the period for small oscillations about $X_0 = -1$, and infinity when the 'amplitude' is 3 and the solution for X_0 is a solitary wave, $1 - 3\,\mathrm{sech}^2\,(\tfrac{1}{2}(\tau + \phi))$. The period tends to infinity because of the asymptotic approach along the homoclinic orbit to the unstable equilibrium point $X_0 = 1$. This means that the two-time approach breaks down as the effective fast timescale becomes longer. However a uniformly valid solution can still be constructed, at least for t not close to $\pm\pi$. This is the n-spike solution discussed by Ockendon *et al.* (1986) where Y_0 remains close to $+\surd 2|\cos\tfrac{1}{2}t|$ over most of the t interval, (t not close to $\pm\pi$) but additionally has n 'spikes' near $t = 0$.

In addition to its failure near the homoclinic orbit the two-time approach can also break down at points where $f'(t)$ equals zero, that is at $t = \pm\pi$ modulo 2π. In this instance it is actually a breakdown of the Kusmak criterion for choosing $f(t)$. This arises because the effective potential changes from (2.12), which has locally quadratic behaviour at the minimum $X_0 = 1$, to one with a stationary point with locally cubic behaviour. Thus with a different local choice of $f(t)$ and a different choice of local scaling of Y_0, we require a solution of

$$X_{0\tau}^2 = \tfrac{1}{3}X_0^3 + \text{constant} \tag{2.13}$$

to find the behaviour of Y_0.

The solutions of this equation are no longer periodic and all blow up in finite time except the trivial solution $X_0 = 0$. Again the asymptotic approach to this solution as $t \to \pm\pi$ implies that the two-time approach has broken down.

We now return to the expansion of (2.6). The next equation comes from the coefficient of ϵ and assuming $c = o(\epsilon)$ gives

$$f'^2 Y_{1\tau\tau} - 2Y_0 Y_1 = -\{f'' Y_{0\tau} + 2f' Y_{0\tau t} + \delta f' Y_{0\tau}\}. \tag{2.14}$$

Since $Y_{0\tau}$ satisfies the homogeneous part of (2.4) we may write

$$-Y_1\{f'^2 Y_{0\tau\tau\tau} - 2Y_0 Y_{0\tau}\} + Y_{0\tau}\{f'^2 Y_{1\tau\tau} - 2Y_0 Y_1\} \equiv [f'^2(Y_{0\tau} Y_{1\tau} - Y_1 Y_{0\tau\tau})\}_\tau = \Theta_{0\tau} G(t, \tau), \tag{2.15}$$

where $G(t, \tau)$ is the right-hand side of (2.14). The condition that Y_1 is periodic in τ requires that $Y_{0\tau}$ is orthogonal to $G(t, \tau)$ over a complete period. Thus

$$\oint (f'' Y_{0\tau} + 2f' Y_{0\tau t} + \delta f' Y_{0\tau}) Y_{0\tau} \, d\tau = 0, \tag{2.16}$$

so that

$$\oint \frac{\partial}{\partial t}\{f' Y_{0\tau}^2 \, e^{\delta t}\} \, d\tau = 0. \tag{2.17}$$

This may be integrated with respect to t to give

$$e^{\delta t}\{f' \oint Y_{0\tau}^2 \, d\tau\} = \text{constant}. \tag{2.18}$$

The term $f' \oint Y_{0\tau}^2 \, d\tau$ is actually independent of the scaling functions f and represents the action. The interpretation of (2.18) is thus that the action is an exponentially decreasing function of time if $\delta > 0$ or constant if $\delta = 0$. The orbit followed by the slowly varying solution in the $(Y_0, Y_{0\tau})$ phase plane comprises an approximate closed loop lying within the homoclinic orbit. The area of this loop is $\oint Y_{0\tau}^2 \, d\tau$ which, by (2.18), increases slowly as f' decreases, whereas by (2.10) the size of the homoclinic orbit clearly decreases as $1 + \cos t \equiv 2^{-\frac{1}{2}} f'$ decreases. Thus the case where the action is of order one must be the n-spike solution, where (2.18) breaks down because the closed loop in the $(Y_0, Y_{0\tau})$ phase plane tends to the homoclinic orbit before $|t|$ reaches π. Similarly if the oscillations are to extend to $t = \pm\pi$ then the action must be zero or at least $o(1)$ as $\epsilon \to 0$. This argument suggests that in this case the appropriate solution is one of small amplitude about the equilibrium, which must still break down at $t \to \pm\pi$, where the two-time approach fails. We shall also require a theory for the 'inner' solution valid near $t = \pm\pi$. The problem of finding a periodic solution to (2.1) thus divides itself into two phases. There is an outer solution valid in $-\pi < t < \pi$. This must be connected by an inner solution, valid near $t = -\pi(\equiv \pi \bmod 2\pi)$. We shall also indicate how to construct a uniformly valid solution for the n-spike solution which requires a third solution connecting the inner and outer solution since the inner solution does not extend to $t = \pm\pi$.

2.2. *The outer solution*

We now look for a series solution of (2.6) but assume that each term is expanded as

$$Y_i = Y_i^0 + \epsilon^p Y_i^1 + \epsilon^{2p} Y_i^2 + \dots, \tag{2.19}$$

where Y_0^0, the equilibrium position, is a function of t only and p is a number to be determined. We shall find in the following section that the critical size of c is $O(\epsilon^{\frac{4}{3}})$ and write for convenience $c = 2^{-\frac{2}{3}} \gamma \epsilon^{\frac{4}{3}}$. This will of course mean that additional terms will be required in the full expansion for Y. These are ignored for the present as they do not affect the leading terms.

Hence we write the full expansion as

$$Y(\tau, t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \epsilon^{np+m} Y^n_m. \tag{2.20}$$

When this is introduced into (2.6), we obtain

$$\begin{aligned}\epsilon^p f'^2 Y^1_{0\tau\tau} + \epsilon^{2p} f'^2 Y^2_{0\tau\tau} + \epsilon^{3p} f'^2 Y^3_{0\tau\tau} + \epsilon^{p+1}\{f'' Y^1_{0\tau} + 2f' Y^1_{0t\tau} + \delta f' Y^1_{0\tau} + f'^2 Y^1_{1\tau\tau} \\ = Y^{02}_0 + 2\epsilon^p Y^0_0 Y^1_0 + \epsilon^{2p}(2 Y^0_0 Y^2_1 + Y^{12}_0) + 2\epsilon^{p+1} Y^0_0 Y^1_1 - 1 - \cos t - 2^{-\frac{2}{3}}\gamma\epsilon^{\frac{4}{3}} \dots\end{aligned} \tag{2.21}$$

The ϵ^0 term simply gives the equilibrium solution

$$Y^{02}_0 = 1 + \cos t = 2\cos^2 \tfrac{1}{2}t, \tag{2.22}$$

or

$$Y^0_0 = \pm\sqrt{2}\cos\tfrac{1}{2}t \quad (-\pi < t < \pi). \tag{2.23}$$

The ϵ^p term gives

$$f'^2 Y^1_{0\tau\tau} - 2Y^0_0 Y^1_0 = 0, \tag{2.24}$$

and the criterion for choosing f outlined in the previous section requires that the solution for Y^1_0 is periodic in τ with period independent of t. Hence we choose

$$f' = \sqrt{2}|Y^0_0|^{\frac{1}{2}} = 2^{\frac{3}{4}}\cos^{\frac{1}{2}}\tfrac{1}{2}t. \tag{2.25}$$

This integrates to give

$$f(t) = 2^{\frac{3}{4}}\int_{-\pi}^{t} \cos^{\frac{1}{2}}\tfrac{1}{2}\xi \, \mathrm{d}\xi, \tag{2.26}$$

the constant of integration being absorbed into τ. Following the discussions at the end of §2.1 if we now take the negative sign in (2.23), (2.24) may be written as

$$Y^1_{0\tau\tau} + Y^1_0 = 0, \tag{2.27}$$

with solution

$$Y^1_0 = A^1_0(t)\cos(\tau + \phi(t)) \equiv A^1_0(t)\cos\theta. \tag{2.28}$$

The functional dependence of A^1_0 and ϕ now follows the usual method of slowly varying amplitude and phase. The equation for A^1_0 is determined by the analogue of (2.15) which comes from the ϵ^{p+1} terms in (2.21). There is normally a second condition that determines ϕ. However, when using Kusmak's method this slow variation can always be incorporated by a suitable expansion of f and, at this order, it suffices to take ϕ to be constant. Thus the ϵ^{p+1} terms give

$$\begin{aligned}|Y^0_0|(Y^1_{1\tau\tau} + Y^1_1) &= -\{f'' Y^1_{0\tau} + 2f' Y^1_{0t\tau} + \delta f' Y^1_{0\tau}\} \\ &= +\{f'' A^1_0 + 2f' A^1_{0t} + \delta A^1_0 f'\}\sin\theta.\end{aligned} \tag{2.29}$$

The resonance condition requires that the coefficient of $\sin\theta$ is zero so that

$$A^1_0 = 2^{-\frac{13}{24}} C\, \mathrm{e}^{-\delta t/2}\cos^{-\frac{1}{4}}\tfrac{1}{2}t, \tag{2.30}$$

where C is a constant.

Thus the first approximation gives

$$Y \approx -\sqrt{2}\cos\tfrac{1}{2}t + \frac{\epsilon^p 2^{-\frac{13}{24}} C\, \mathrm{e}^{-\delta t/2}}{\cos^{\frac{1}{4}}\frac{1}{2}t}\cos(\tau + \phi). \tag{2.31}$$

We anticipate the results of §2.4 (see (2.48)) which requires that $p = \frac{5}{6}$ and look at the solution near $-\pi$ by writing $t = -\pi + 2^{\frac{1}{6}}\epsilon^{\frac{2}{3}} T$. Then

$$\tau = \frac{2^{\frac{3}{4}}}{\epsilon}\int_{-\pi}^{t}\cos^{\frac{1}{2}}\xi\, \mathrm{d}\xi = \frac{2^{\frac{3}{4}}}{\epsilon}\int_0^T 2^{\frac{1}{6}}\epsilon^{\frac{2}{3}}\sin^{\frac{1}{2}}(\tfrac{1}{2}2^{\frac{1}{6}}\epsilon^{\frac{2}{3}}\eta)\, \mathrm{d}\eta \sim \frac{2\sqrt{2}}{3}T^{\frac{3}{2}}, \tag{2.32}$$

and the solution for y becomes

$$2^{\frac{1}{3}}\epsilon^{-\frac{2}{3}}y \approx -T+\frac{\epsilon^{p-\frac{5}{6}}C\mathrm{e}^{\pi\delta/2}}{T^{\frac{1}{4}}}\cos\left(\frac{2\sqrt{2}T^{\frac{3}{2}}}{3}+\phi\right). \tag{2.33}$$

and similarly, near $t=\pi$ with $t=\pi+2^{\frac{1}{3}}\epsilon^{\frac{2}{3}}T$ with $T<0$, we obtain

$$\tau=\frac{I}{\epsilon}-\frac{2\sqrt{2}}{3}|T|^{\frac{3}{2}}, \tag{2.34}$$

and

$$2^{\frac{1}{3}}\epsilon^{-\frac{2}{3}}y \approx |T|+\frac{\epsilon^{p-\frac{5}{6}}}{|T|^{\frac{1}{4}}}C'\mathrm{e}^{-\pi\delta/2}\cos\left(\frac{I}{\epsilon}-\frac{2\sqrt{2}}{3}|T|^{\frac{3}{2}}+\phi\right), \tag{2.35}$$

where

$$I=2^{\frac{3}{4}}\int_{-\pi}^{\pi}\cos^{\frac{1}{2}}\tfrac{1}{2}\xi\,\mathrm{d}\xi. \tag{2.36}$$

Since I/ϵ equals the integral $\int_{-\pi}^{\pi}\tau\,\mathrm{d}t$ this also represents the total change of phase in the outer solution. If n is the nearest integer to $I/(2\pi\epsilon)$ then n represents the number of complete cycles or spikes in the outer solution.

2.3. *The inner solution*

The outer solution breaks down for two reasons. First, as $t\to\pm\pi$ the ratio of successive terms in the expansion of Y_0 in powers of ϵ^p becomes large. Second the derivatives of Y_0^0 with respect to t become large so that we anticipate that $\ddot{y}$ is becoming large.

Since Y_0^0 is zero at $t=-\pi$ we introduce scaled variables

$$x=\epsilon^{-q}y,\quad T=(t+\pi)\,\epsilon^{-r}. \tag{2.37}$$

Then we may rewrite (2.1) as

$$\epsilon^{2+q-2r}\{x''+\delta\epsilon^r x'\}=\epsilon^{2q}x^2-\tfrac{1}{2}\epsilon^{2r}T^2-c+O(\epsilon^{4r}). \tag{2.38}$$

The leading terms from $\ddot{y}$, y^2 and $\cos t+1$, are of the same order if

$$2+q-2r=2q=2r\quad\text{or}\quad q=r=\tfrac{2}{3}. \tag{2.39}$$

This is a consistent scaling if c is of order $\epsilon^{\frac{4}{3}}$ or smaller. If c is larger, say of order ϵ^n with $n<\frac{4}{3}$, then the leading term from $1+\cos t$ is of smaller order and we must choose

$$q=\tfrac{1}{2}n,\quad r=1-\tfrac{1}{4}n\quad(>\tfrac{2}{3}). \tag{2.40}$$

In this case the behaviour of the inner solution is governed by the ordinary differential equation

$$x''+\delta\epsilon^r x'=x^2-c^*, \tag{2.41}$$

where c^* is c/ϵ^n.

This equation represents damped cnoidal waves or periodic waves if $\delta=0$. If this is to be properly matched with the outer solution then the neglected terms in the expansion of (2.20) which involve c must be included. These appear in the expansion of f to modify the choice of f' so that it has a small but positive minimum. An alternative explanation is that one has to reconstruct the expansion of $Y_0=\pm(1+\cos t+c)^{\frac{1}{2}}$ in order to obtain a uniformly valid expansion. We do not discuss these details here as we merely obtain the results of Ockendon *et al.* (1986) in the limit of small but order-one values of c.

Hence the most general case of interest is c of order $\epsilon^{\frac{4}{3}}$. For convenience we introduce $\gamma = 2^{\frac{2}{3}} c \epsilon^{-\frac{4}{3}}$ and with $q = r = \frac{2}{3}$ we modify (2.37) and write the scaled variables as

$$x = 2^{\frac{1}{3}} \epsilon^{-\frac{2}{3}} y, \quad T = 2^{-\frac{1}{6}} \epsilon^{-\frac{2}{3}}(t+\pi). \tag{2.42}$$

If we expand x in a power series

$$x = x_0 + \epsilon^{\frac{2}{3}} x_1 + \dots, \tag{2.43}$$

then the first-order inner problem from (2.1) is

$$x_0'' = x_0^2 - T^2 - \gamma. \tag{2.44}$$

In order to match to the outer solution, we require the asymptotic form of this equation valid as $T \to +\infty$. Assuming the existence of solutions of (2.44) that remain bounded for all finite T, the leading term is $\pm T$ as $T \to \mp \infty$. Noting that large positive T implies t tends to $-\pi$, in order to match with (2.33) we require the minus sign and there are two types of asymptotic behaviour, at least for $\gamma \neq 0$. The first we may write as

$$x_0 = \bar{x}_0(T) \sim -T - \frac{\gamma}{2T} - \frac{\gamma^2}{8T^3} + \dots, \tag{2.45}$$

The right-hand side being a series in powers of T^{-1}. The second type of asymptotic form has oscillatory behaviour about $\bar{x}_0(T)$ with an algebraic decay that is slower than (2.45). The leading terms are

$$x_0 \sim -T + \frac{A_+}{T^{\frac{1}{4}}} \cos\left(\frac{2\sqrt{2}}{3} T^{\frac{3}{2}} + \phi_+\right). \tag{2.46}$$

Similarly as $T \to -\infty$ we have

$$x_0 \sim -|T| + \frac{A_-}{|T|^{+\frac{1}{4}}} \cos\left(\frac{2\sqrt{2}}{3} |T|^{\frac{3}{2}} + \phi_-\right). \tag{2.47}$$

2.4. *Matching and the connection problem for the periodic solution*

We now look in detail at the matching required to produce a 2π-periodic solution of (2.1). Comparing (2.46) and (2.33) we see that an outer solution of the form (2.31) matches an inner solution of (2.44) near $t = -\pi$ if

$$2^{\frac{1}{3}} \epsilon^{-\frac{2}{3}} y(T) = x_0(T), \tag{2.48}$$

where $x_0(T)$ satisfies (2.46). This requires, as we have already stated, $p = \frac{5}{6}$ and that the following conditions are satisfied:

$$A_+ = C\,\mathrm{e}^{\pi\delta/2}, \quad \phi_+ = \phi. \tag{2.49}$$

As $t \to \pi$ from below, y satisfies the asymptotic form (2.35) and since y is periodic this is also the asymptotic form of the outer solution for $t \to -\pi$ from below. Hence comparing (2.35) and (2.47) we obtain two further conditions:

$$A_- = C\,\mathrm{e}^{-\pi\delta/2}, \quad 2n\pi - \phi_- = I/\epsilon + \phi. \tag{2.50}$$

We may combine (2.49) and (2.50) to give

$$A_+ = A_-\,\mathrm{e}^{\pi\delta}, \quad I/\epsilon = 2n\pi - \Delta\phi, \tag{2.51}$$

where

$$\Delta\phi = \phi_+ + \phi_-. \tag{2.52}$$

The two conditions of (2.51) define a connection problem which must be satisfied if the solution x_0 of (2.44) is to give a 2π-periodic solution of (2.3). This problem may be reformulated in the following way. Suppose x_0 is the solution of (2.44) with asymptotic forms (2.46) and (2.47) as $T \to \pm\infty$ and suppose also that $x_0(0) = \alpha$ and $\dot{x}_0(0) = \beta$. Then for $T > 0$ we write $x_0(T) = x_+(T)$ and for $T < 0$ we write $x_0(-T) = +x_-(T)$. Then x_+ and x_- both satisfy

$$x_0'' = x_0^2 - T^2 - \gamma, \quad T > 0, \tag{2.53}$$

with initial conditions

$$x_+(0) = \alpha, \quad \dot{x}_+(0) = \beta, \tag{2.54}$$

$$x_-(0) = \alpha, \quad \dot{x}_-(0) = -\beta. \tag{2.55}$$

The advantages of the transformation to x_- for $T < 0$ is that now x_- satisfies (2.46) but with (A_+, ϕ_+) replaced by (A_-, ϕ_-).

For a fixed value of $\Delta\phi$ in the range, say, $-2\pi < \Delta\phi \leqslant 0$, we have to find α and β that satisfy the two conditions

$$A_+ = A_- \,\mathrm{e}^{\pi\delta}, \quad \Delta\phi = \phi_+ + \phi_-. \tag{2.56}$$

In practice, rather than fix $\Delta\phi$ and iterate, one fixes, say, β, and finds the value of α that satisfies the first of (2.56) and merely calculates $\Delta\phi$ as a function of β from the solutions.

In this form the connection problem is seen to be independent of ϵ. After the values of α and β, which allow a solution of (2.56), have been found, the second condition of (2.51) is used to determine which value of ϵ this solution corresponds to. The role of n appearing in (2.51) now becomes apparent. The phase functions ϕ_+ and ϕ_- are only defined modulo 2π. If they are chosen so that $-2\pi < \Delta\phi \leqslant 0$ then for a given solution there will be a sequence of values $\epsilon(n)$, $n = 0, 1, \ldots$ that monotonially decreases as n increases. Each succeeding value of ϵ corresponds to a solution with an extra oscillation in the outer solution.

3. Discussion of the solutions

This is the case of zero damping, corresponding to inviscid flow. From (2.51) this requires A_+ to be equal to A_-. We term this common value A. The solution of (2.53) is even if $\dot{x}(0) = 0$ and this clearly gives a symmetric solution with $A_+ = A_- \equiv A$. Thus the required solution is $\beta = 0$ with $x_+ \equiv x_-$. This in fact proves to be the only set of solutions with $A_+ = A_-$. Hence we solve with $\beta = 0$ and simply compute $\Delta\phi(\alpha)$. In the (α, β) phase plane of initial conditions, the locus lies along the α-axis but is restricted to $\alpha_- \leqslant \alpha \leqslant \alpha_+$ where $\alpha_-(\gamma)$ and $\alpha_+(\gamma)$ are functions of γ. Outside this interval no solutions of the type sought exist. Ockendon *et al.* (1986) and Byatt-Smith (1988) give numerical evidence to show that when $\gamma = -1.466\ldots$, $\alpha_- = \alpha_+$ and as γ increases the range $\alpha_+ - \alpha_-$ increases and tends to infinity as $\gamma \to +\infty$. The graph of $\phi_+ = \frac{1}{2}\Delta\phi$ is shown in figure 1 plotted as a function of the initial condition α, for the case $\gamma = 0$. The phase ϕ_+ has a maximum and tends to $-\infty$ as $\alpha \to \alpha_+$ or α_-. (Here of course we have taken $\Delta\phi$ to be defined continuously rather than restricted to values in the range $-2\pi < \phi \leqslant 0$.) Following (2.51) this means that for a given value of n there is a maximum value of ϵ beyond which the solution fails to exist. The corresponding diagram for A, figure 2, shows that A has a minimum and tends to ∞ as $\alpha \to \alpha_+$ or α_-. This is also shown in figure 3 where the allowable values of (A_+, ϕ_+) are plotted as a curve parameterized by $\alpha \in [\alpha_-, \alpha_+]$ in the (A, ϕ)-plane. Although the process of constructing a solution breaks down if A becomes too large, the

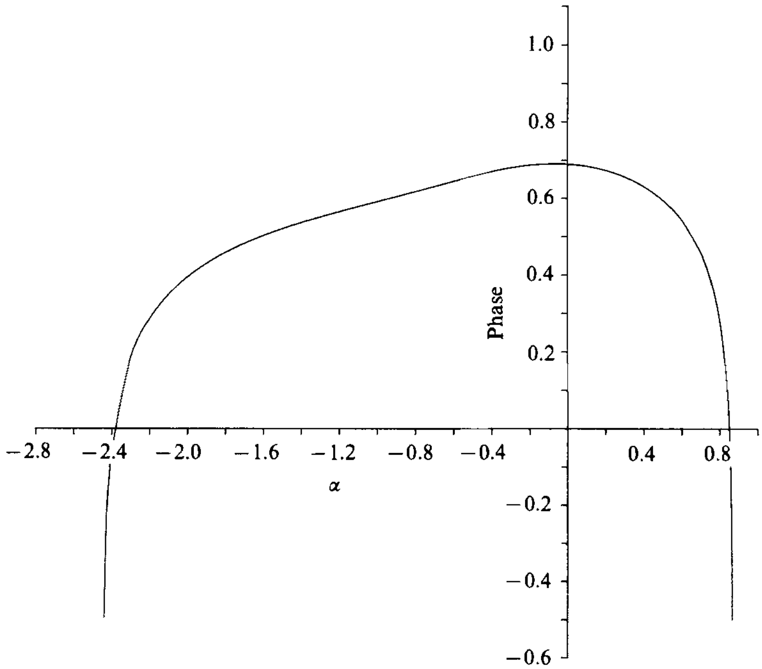

FIGURE 1. The graph of $\phi_+(\alpha)$ for the symmetric solution when $\gamma = 0$. In this diagram and all subsequent diagrams involving ϕ the variable ϕ has been scaled so that the plotted variable equals the variable in the text divided by 2π

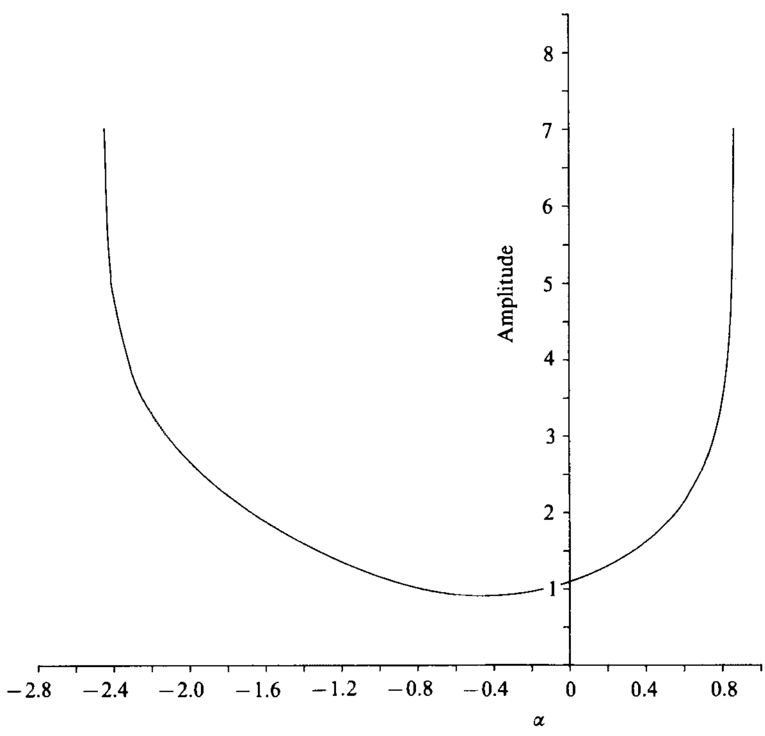

FIGURE 2. The graph of $A_+(\alpha)$ for the symmetric solution when $\gamma = 0$.

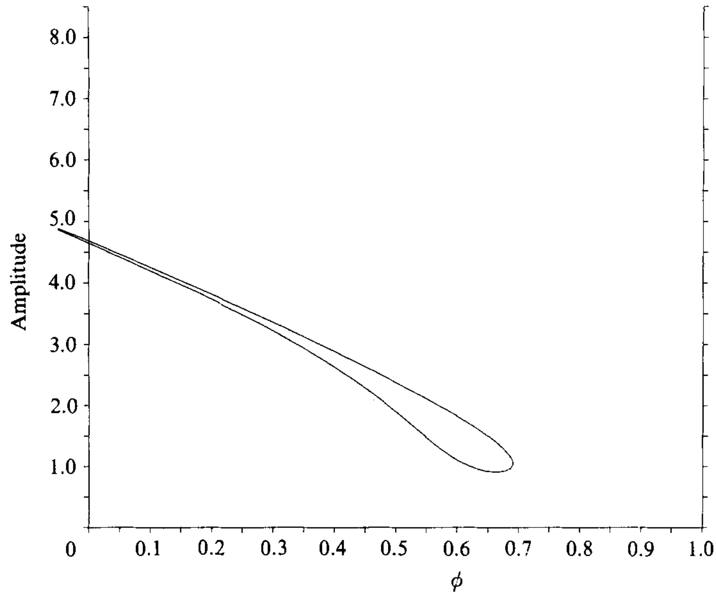

FIGURE 3. The curve $(A_+, \phi_+) \equiv \Gamma_0$ for the symmetric solution when $\gamma = 0$ plotted in the (A, ϕ)-plane.

limiting solutions for the inner problem when $\alpha = \alpha_+$ and $\alpha = \alpha_-$ can be shown to be those that tend to $x_0 = |T|$ as $T \rightarrow \pm\infty$. Holmes (1982) proves that when $\gamma = 0$ there is a unique solution of (2.44) with $\alpha \equiv x_0(0) = \alpha_+ > 0$ and a unique solution with $\alpha \equiv x_0(0) = \alpha_- < 0$ that satisfies the requirement that $x_0 \sim +|T|$ as $T \rightarrow \pm\infty$. These are even solutions and represent the $j = 0$ or 1 'spikes' in the solution at $t = \pi$ (see §1). The graphs of these functions (for $\gamma = 0$) are shown in Holmes (1982, figure 1) and Ockendon *et al.* (1986, figure 6). Byatt-Smith (1988) demonstrates numerically that there continue to be two and only two values of α that give solutions of (2.44) with $x_0 \sim +|T|$ provided $\gamma > -1.466\ldots$, the value at which the two solutions merge, that is when $\alpha_+ = \alpha_-$. The convergence to the limiting inner solution as $\alpha \rightarrow \alpha_+$ from below or α_- from above is only pointwise in T and not uniform for all T in $[0, \infty)$. The role of the limiting solution and the breakdown of the inner solution due to the non-uniform convergence is discussed in the next section.

The limiting solutions have to be matched to an outer solution with $Y_0^0 > 0$. The analogue of (2.31) that is bounded in $-\pi < t < \pi$ is

$$y \approx +\sqrt{2}\cos\tfrac{1}{2}t + \frac{\epsilon^p 2^{-\frac{13}{24}} C^*}{\cos^{\frac{1}{4}}(\frac{1}{2}t)} \frac{\cosh(\tau - I/(2\epsilon))}{\cosh(I/(2\epsilon))}. \tag{3.1}$$

Again near $t = \pm\pi$ we can obtain the asymptotic result

$$2^{\frac{1}{3}} \epsilon^{-\frac{2}{3}} y = +|T| + \frac{\epsilon^{p-\frac{5}{6}}}{|T|^{\frac{1}{4}}} C^* \exp\left(\frac{-2\sqrt{2}}{3}|T|^{\frac{3}{2}}\right), \tag{3.2}$$

and matching can be achieved provided the value of C^* can be obtained from the inner solution. However the mathematical and numerical processes involved in the matching procedure are much more complicated in this case. This is because over most of the range of t the second term in (3.1) is exponentially smaller than some of

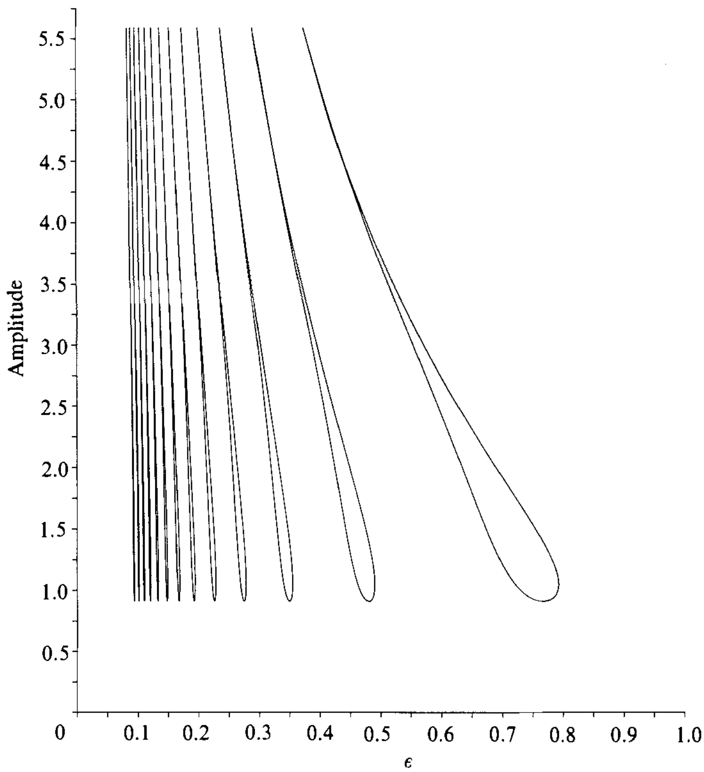

FIGURE 4. The amplitude response curve for $\gamma = 0$.

those that arise from the higher-order terms in (2.21). Thus (3.1) represents the leading terms in the series (2.20) which are technically of order 1 and ϵ^p, but the second term is actually only of order ϵ^p when τ is $O(1)$ or $\tau - I/(2\epsilon)$ is $O(1)$. This is of course not compatible with $|t|-\pi$ of order 1. The higher-order terms fall into the same categories: either they are independent of τ, like Y_0^0 or they have exponential behaviour in τ similar to that observed in Y_0^1. Thus outside the ranges $\tau = O(1)$ and $\tau - I/(2\epsilon) = O(1)$ these terms are smaller than any of the terms that are independent of τ but grow exponentially rapidly as we approach these ranges of τ which correspond to $|t|-\pi = O(\epsilon^{\frac{2}{3}})$. This is reflected by the fact that the decay to $|T|$ in the inner solution is algebraic, in contrast to (3.2). In more detail the inner solution has an asymptotic behaviour, equation (3.3), which is analogous to (2.46). In (3.3) $x_0^*(T)$ is the particular function that corresponds to $\bar{x}_0(T)$ in (2.45) but now tends asymptotically to $+|T|$ as $T \to \pm\infty$, and differs from $|T|$ only algebraically. The analogue of (2.46) is then

$$x_0 = x_0^*(T) + \frac{A_+}{T^{\frac{1}{4}}} \exp\left(-\frac{2\sqrt{2}}{3} T^{\frac{3}{2}}\right) \quad \text{as } T \to +\infty. \tag{3.3}$$

Since $x_0^*(T)$ is not known in closed form the numerical computation of A_+ is not straightforward. The matching condition that we wish to invoke is that the algebraic terms from $x_0^*(T)$ will match the higher-order terms in (2.20) that are independent of τ, while the exponential terms of (3.3) will match the terms that have exponential behaviour in τ. The first condition is automatically satisfied (that is $2^{\frac{1}{3}}\epsilon^{-\frac{2}{3}} y = |T| = x_0^*(T)$ to leading order) while the second condition determines C^*. When γ is

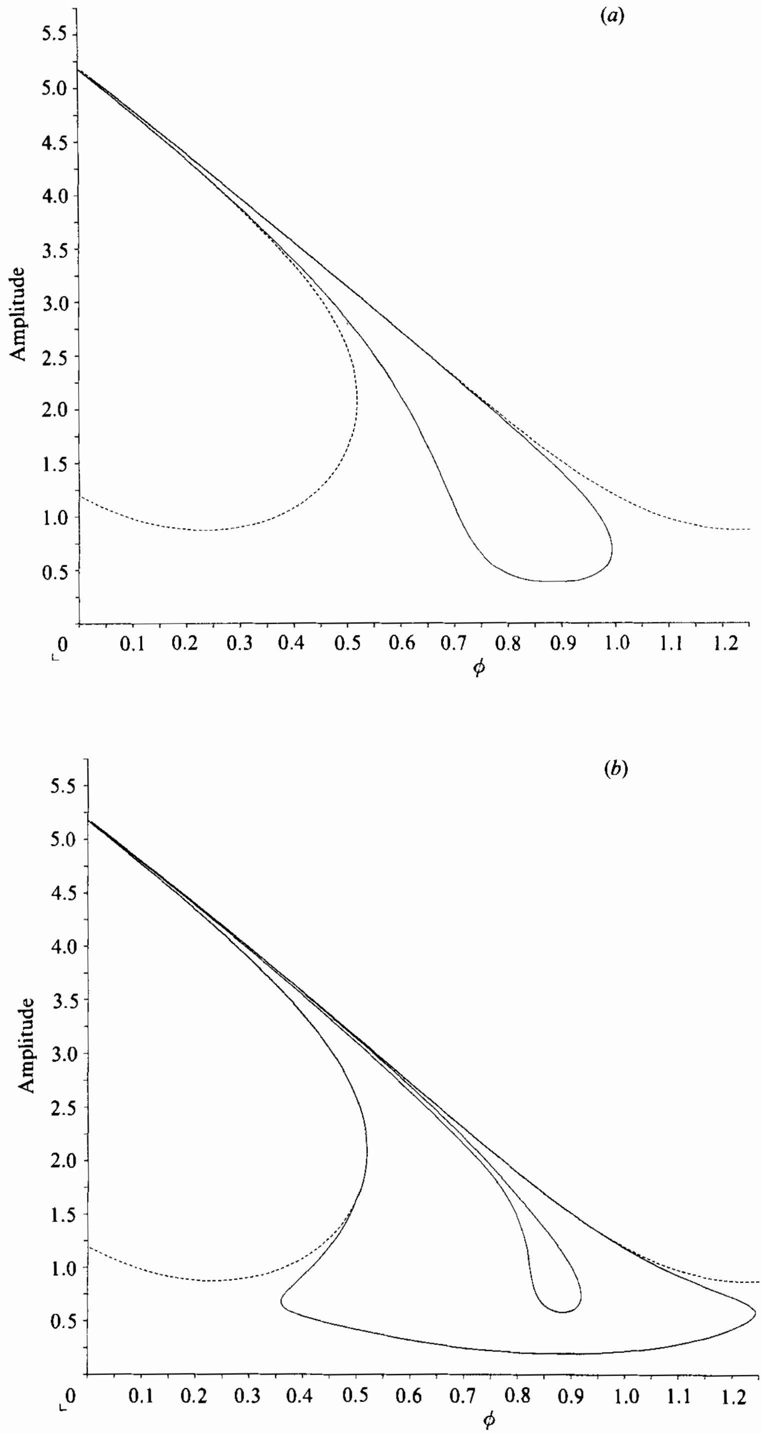

FIGURE 5(a, b). For caption see facing page.

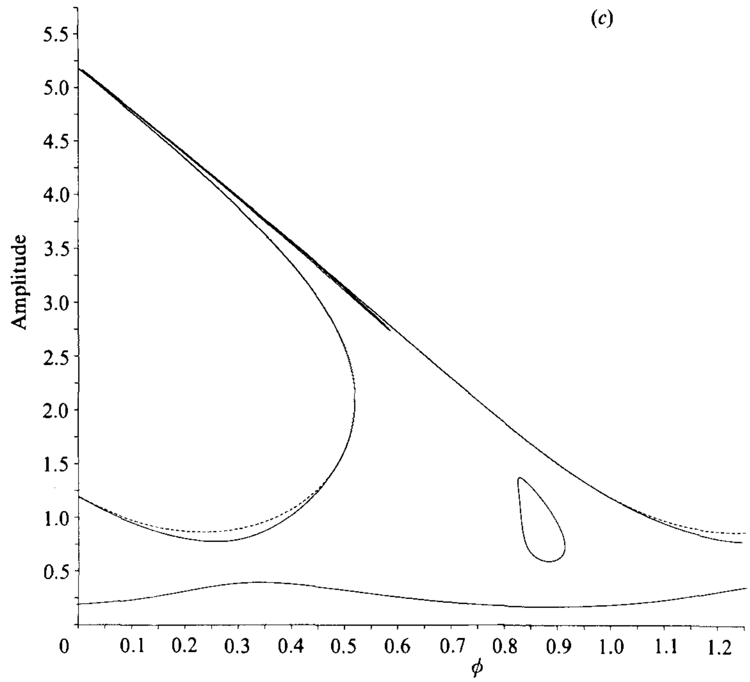

FIGURE 5. (*a*) The curve Γ_0 when $\gamma = 1$. The dotted curve represents the boundary D. (*b*) The curves Γ_+ and Γ_- for $\gamma = 1$ when $\Delta = 3$; again the dotted curve represents the boundary D. (*c*) The curves Γ_+, Γ_- for $\gamma = 1$, $\Delta = 3.5$ and the boundary D.

identically zero there are no algebraic terms in (2.45) and $x_0^*(T)$ is identically equal to $|T|$ and the numerical computation of A_+ is then considerably easier.

Once $(A(\alpha), \phi(\alpha))$ have been determined the second condition of (2.51) determines the value of ϵ that this solution corresponds to. This is shown in figure 4 with A plotted as a function of ϵ. The value of ϕ is defined continuously along these curves and the value of n is determined by the condition that the maximum value of ϕ lies between 0 and 2π. The curves shown are for values of n between 3 and 14. The curve corresponding to $n = 2$ has a maximum value of $\epsilon > 1$ while that corresponding to $n = 1$ tends to infinity. This is because the maximum value of ϕ is greater than π, so that $\Delta\phi = 2\pi$ at some point along the (A, ϕ)-curve of figure 3. It should be remembered however that the theory is only valid for small values of ϵ. The curves corresponding to $\gamma = 1$ are shown in figures 5(*a*) and 6(*a*).

In any practical situation the quantity ϵ is fixed and the quantity γ is varied. The quantity c from which γ is defined is simply the average

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} y^2\,\mathrm{d}t - 1$$

at least for periodic solutions. The response diagram for equation (2.1) is shown in figure 7 with γ plotted as a function of amplitude A_+ for $\epsilon = 0.1$. The curves that are shown are $n = 16(1)12$. Each curve has a minimum value of γ and a minimum value of A_+. The minimum value of γ decreases as n decreases and the curve corresponding to $n = 12$ has a minimum value at $\gamma \approx -1.451$, which is only just greater than the

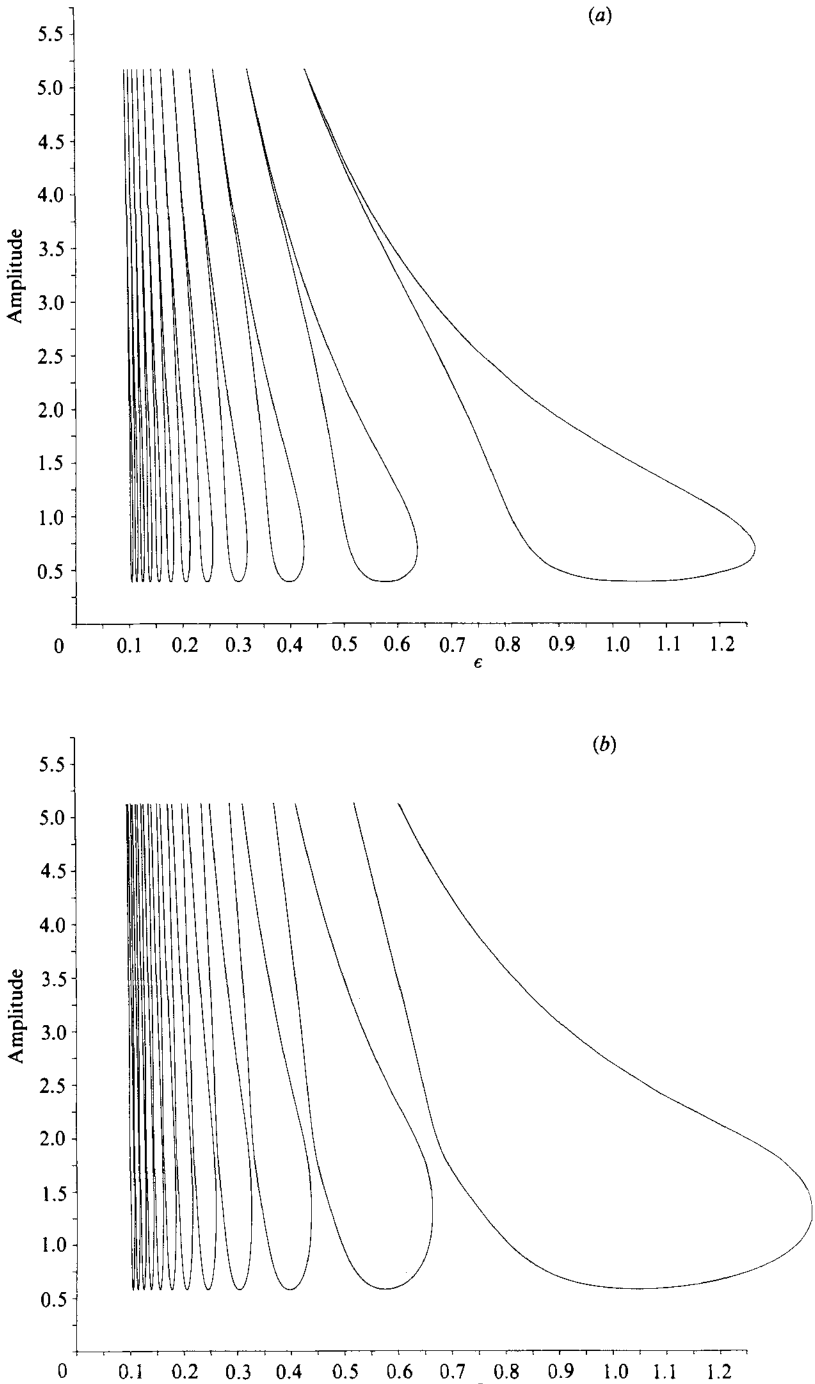

FIGURE 6 $(a,\, b)$. For caption see facing page.

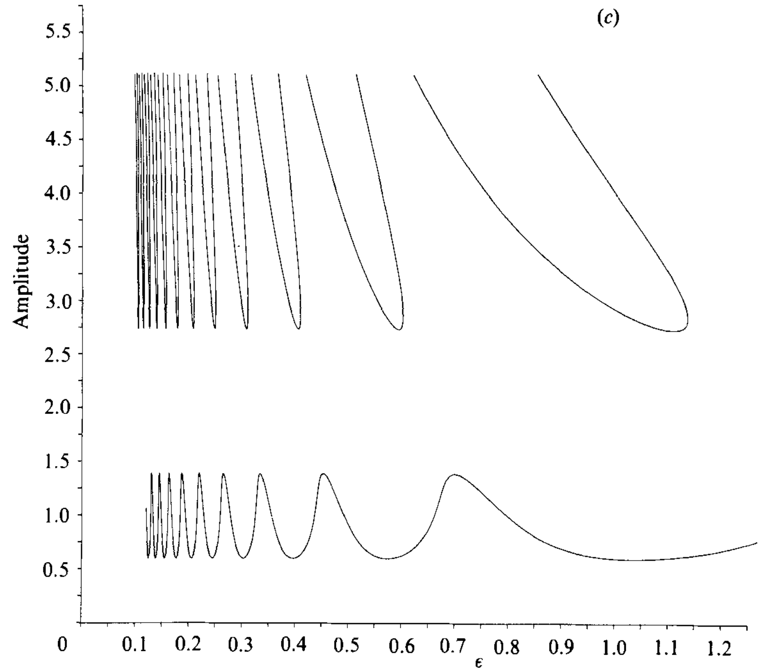

FIGURE 6(a–c). The amplitude response curves corresponding to figure 5(a–c).

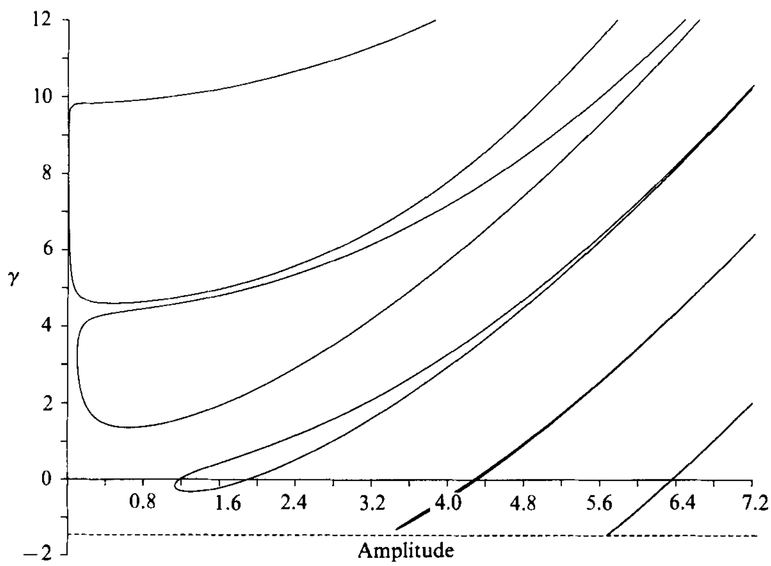

FIGURE 7. The γ–A response curve for $\epsilon = 0.1$ $\Delta = 1$ ($\delta = 0$). The dotted line is $\gamma = -1.466\ldots$.

minimum value $-1.466\ldots$ beyond which the bounded solutions cease to exist. This value of γ is shown by the dotted line.

For smaller values of n the values of (A_+, ϕ_+) effectively satisfy the asymptotic result of equation (3.12) (see next section). For a fixed value of γ we label the A_+ values as A_{+n}. The corresponding values of ϕ_{+n}, if defined continuously, must increase by π as n increases by 1. The alternative description is to define the two values of $\phi = \phi_{+n_0}$ and $\phi = \phi_{+n_0+1}$, in $(0, 2\pi)$ that correspond to the given value of ϵ. Then, regarding the (A, ϕ)-diagram of figure 3 (or the appropriate one for the value of γ chosen) as a cylinder with $\phi = 0$ and $\phi = 2\pi$ identified, the $A_+(\alpha)$, $\phi_+(\alpha)$ curve winds repeatedly around the cylinder with $A_+(\alpha)$ monotonically increasing. Then A_{+n_0-2i} and A_{+n_0-2i+1} are the successive intersections of the curve with the lines $\phi = \phi_{+n_0}$ and $\phi = \phi_{+n_0+1}$.

Using (3.12) we may then obtain

$$A^{\frac{6}{5}}_{n-1} - A^{\frac{6}{5}}_{n} = -k_1(\phi_{+n-1} - \phi_{+n}) = \pi k_1, \tag{3.4}$$

or

$$A^{\frac{6}{5}}_{n-1}(\gamma) = A^{\frac{6}{5}}_{n}(\gamma) + \pi k_1, \tag{3.5}$$

For larger values of n, the minimum value of A_+ becomes very small. Numerical evidence suggests that this minimum is exponentially small for large values of γ. For these larger values of n, the response diagram has the familiar resonance horns that occur for forced second-order differential equations (see for example Hale & Rodriques 1977*a*, *b*; Holmes & Rand 1976 and Guambaudo 1985). These resonance horns branch off the $A = 0$ axis over relatively short intervals of the γ-axis. Numerical evidence suggests that the centres of these intervals on the γ-axis increase as $n^{\frac{4}{3}}$, so that successive intervals on the γ-axis where A is exponentially small have a width of order $n^{\frac{1}{3}}$.

This can be explained by looking at the behaviour of the (A_+, ϕ_+) or Γ_0 curve for $\delta = 0$, as γ increases. A comparison between figures 3 and 5(*a*) shows that Γ_0 has a maximum value of ϕ_+ which increases and a minimum value of A_+ which decreases as γ has increased from 0 to 1. This trend continues and for large values of γ the curve has a very flat minimum with $A_+ \approx 0$ over an interval of ϕ_+ values that is approximately π in width and that moves to the right with increasing γ. At either end of this range of ϕ_+ values, for fixed values of γ, the value of A_+ increases very rapidly.

To obtain the response curve for a fixed value of ϵ, we examine the behaviour of Γ_0 as γ is varied continuously. Through (2.51) and the relation $\phi_+ = \frac{1}{2}\Delta\phi$ a given value of ϵ defines a countable set of possible phases given by

$$\phi_+(n) = n\pi - I/(2\epsilon), \quad n = 0, 1, 2, \ldots, \tag{3.6}$$

where n is restricted to non-negative values by physical considerations. These phases are represented by uniformly spaced lines in the (A, ϕ)-plane. From the general shape of Γ_0 (figures 3 and 5*a*) at a fixed value of γ there exists an integer $N(\gamma)$ such that Γ_0 meets $\phi = \phi_+(n)$ at amplitudes $A^1_n(\gamma)$ and $A^2_n(\gamma)$ for all $n \leqslant N(\gamma)$. Since Γ_0 moves to the right as γ increases, $N(\gamma)$ increases with γ. Thus there exists an increasing sequence γ_n such that $N(\gamma)$ increases by one as γ increases through γ_n. At this value of γ the maximum value of ϕ_+ on Γ_0 is $\phi = \phi_+(n)$. Thus $A^1_n(\gamma)$ and $A^2_n(\gamma)$ coalesce at $\gamma = \gamma_n$ and vary continuously as γ increases beyond γ_n. Thus for each n the union of the two curves $(A^1_n(\gamma), \gamma)$ and $(A^2_n(\gamma), \gamma)$ forms a continuous curve in the (A, γ)-plane, as shown in figure 7, and whose general shape follows readily from the above considerations.

The more conventional response diagram is to plot c, or in this case γ, against the detuning parameter $\hat{\lambda}$. The unperturbed value λ_* from (1.6) can be seen to be

$$\lambda_* = \int_{-\pi}^{\pi} Y^0_0 \,\mathrm{d}t + 2\pi\epsilon^2 = -4\surd 2 + 2\pi\epsilon^2, \tag{3.7}$$

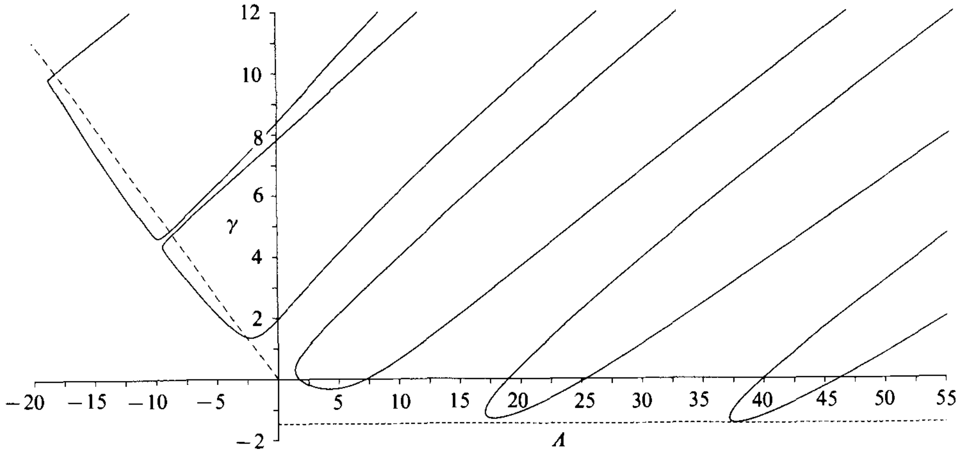

FIGURE 8. The $\gamma - \Lambda$ response curve for $\epsilon = 0.1$ $\Delta = 1$. The second dotted line is $\Lambda = -2^{-\frac{1}{6}}/3\gamma \log(2^{\frac{5}{2}}\epsilon^{-2})$.

and in the Appendix it is shown that

$$\hat{\lambda} = \lambda_* + \epsilon^{\frac{4}{3}}\{2^{-\frac{1}{6}} I(\epsilon) - \tfrac{1}{3} 2^{-\frac{1}{6}} \gamma \log(2^{\frac{5}{2}}\epsilon^{-2})\} + o(\epsilon^{\frac{4}{3}}), \tag{3.8}$$

where $I(\epsilon)$ is an integral evaluated numerically.

For the case of zero damping the connection problem is symmetric so that $I(\epsilon)$ may be written as

$$I(\epsilon) = 2\int_0^{\alpha} (x_0(T) + T)\,\mathrm{d}t + 2\int_{\alpha}^{\infty} (x_0(T) + T + \gamma/2T)\,\mathrm{d}t + \gamma \log \alpha. \tag{3.9}$$

Thus we may define a scaled detuning parameter

$$\Lambda = 2^{-\frac{1}{6}} I(\epsilon) - \frac{2^{-\frac{1}{6}}}{3}\gamma \log(2^{\frac{5}{2}}\epsilon^{-2}), \tag{3.10}$$

so that

$$\hat{\lambda} = \lambda_* + \epsilon^{\frac{4}{3}}\Lambda + o(\epsilon^{\frac{4}{3}}). \tag{3.11}$$

Figure 8 is then a graph of $\gamma(\Lambda)$ for $\epsilon = 0.1$. The second term of (3.10) for fixed ϵ is just a multiple of γ and is independent of the oscillatory perturbation. The first term, while being mathematically of smaller order, is actually larger at this value of ϵ except for those parts of the curves that correspond to small values of Λ. This is of course fairly common when log terms are involved. A comparison between figures 7 and 8 shows that the effect of using Λ as the independent parameter is to open out the graphs at the larger values of γ. Since, in figure 7, the two values of A_+ on a curve for a given value of n at a fixed large value of γ are very close, the difference that occurs in the corresponding Λ values can be attributed to the spike in the connection problem which leads to a lower value of $\int_{-\pi}^{\pi} y\,\mathrm{d}t$.

3.2. *The breakdown of the inner solution*

Before discussing the damped case we look in more detail at the breakdown of the inner solution and the role of the n-spike solution. Byatt-Smith (1988) shows that any set of solutions to (2.44) that converge pointwise to $x = +T$ as $T \to +\infty$ has values (A_+, ϕ_+) that satisfy $\phi_+ \to -\infty$ and

$$A^{\frac{6}{5}} + k_1\phi_+ = k_2 + \gamma k_3 \phi_+^{-\frac{1}{3}} + \dots \text{ as } \phi_+ \to -\infty, \tag{3.12}$$

where k_1, k_2 and k_3 are constants independent of γ. Byatt-Smith (1988) also shows that the first maximum of $x(T)$ in $T > 0$ occurs at a value of T that is of order $|\phi_+|$. Now for sensible scaling problems the inner solution must have reached its asymptotic value on a timescale T that is no greater than $O(\epsilon^{-\frac{2}{3}})$. Thus for small (order-one) values of ϕ_+ and ϕ_-, the role of the inner solution is to connect the outer solution across $T = \pi$ (modulo 2π). However, as ϕ_+ becomes larger the value where oscillations first start in the inner solution (for $T > 0$) becomes larger, so effectively the solution emerges from the inner layer along the curve $x_0 \equiv +2^{\frac{1}{3}}\epsilon^{-\frac{2}{3}}|Y_0^0(t)|$ and the oscillations start in the outer solution at a value of t such that $|t+\pi|$ is greater than $O(\epsilon^{\frac{2}{3}})$. Hence as $\phi_+ \to -\infty$ the solution merges into the n-spike solution.

Thus the construction of a uniformly valid solution then requires three parts: (i) an outer solution of the form (2.20) that has the asymptotic form (3.1) and is valid in $-\pi < t < t_0$ and $2\pi - t_0 < t < \pi$; (ii) an outer solution that satisfies (2.8) in $t_0 < t < 2\pi - t_0$ and also tends to the homoclinic orbit as $t \to t_0$ or $2\pi - t_0$; and (iii) an inner solution with asymptotic form (3.3) valid through $t = -\pi$. This of course corresponds exactly with the construction of Ockendon *et al.* (1986).

3.3. *The case* $\delta > 0$

This is the case of non-zero damping. We now require an asymmetric solution of (2.53) with $A_+ = A_-\,e^{\pi\delta} \equiv \Delta A_-$. The numerical study of Byatt-Smith (1988) shows that for $\gamma > -1.466\ldots$ there is a domain, $D(\gamma)$ in the (A, ϕ) plane such that for any point $(A_-, \phi_-) \in D(\gamma)$ there exists a second point $(A_+, \phi_+) \in D(\gamma)$ and a solution of (2.53) that connects the asymptotic form (2.47) with the asymptotic form (2.46). For fixed δ (in addition to fixed γ) the first condition of (2.51) restricts solution pairs (A_-, ϕ_-), (A_+, ϕ_+) to curves Γ_- and $\Gamma_+ \subset D(\gamma)$. Each point on these curves defines a value of ϵ by means of the second condition of (2.51). For small values of δ the two curves lie close to the single curve Γ_0, corresponding to $\delta = 0$. (This curve, for $\gamma = 0$ is shown in figure 3). The curve Γ_+ lies inside Γ_0, and Γ_- lies outside and all three have the same asymptotic properties as A tends to infinity. (Compare for example figure 5*a*–*c*.)

As δ increases, the curves Γ_- and Γ_+ have two different types of behaviour. When $\gamma \leqslant 0$ the curve Γ_- expands and tends uniformly to the boundary $D(\gamma)$ while Γ_+ contracts and tends to infinity, as δ tends to infinity. This is because for any value of ϕ the point $(0, \phi)$ lies outside $D(\gamma)$, in the case $\gamma < 0$, or possibly on $D(\gamma)$ if $\gamma = 0$. When $\gamma > 0$ all points $(0, \phi)$ $0 \leqslant \phi \leqslant 2\pi$ lie within $D(\gamma)$. All of these points are connected via a solution of (2.53) to a unique point $(A_s(\gamma), \phi_s(\gamma))$. Hence in the limit as $\delta \to \infty$ the ϕ-axis belongs to the curve Γ_- and the point (A_s, ϕ_s) belongs to the curve Γ_+.

This limiting behaviour is approached in the following manner. As δ increases the curve Γ_- still expands, at least initially, until at a critical value of $\delta(\gamma)$ the curve Γ_- touches itself. This happens because Γ_- expands so that there are two points on Γ_- with the same value of A whose ϕ values differ by 2π. As δ increases beyond this critical value Γ_- splits into two parts. One part expands and tends uniformly to the boundary $D(\gamma)$ while the other part tends uniformly to the axis $A = 0$. This latter part of the curve is closed if the points (A, ϕ) and $(A, \phi + 2\pi)$ are identified. The curve Γ_+ still contracts but now splits into two parts at the same critical value of δ. One part tends to infinity while the other is a closed curve which contracts uniformly to the point (A_s, ϕ_s) (see Byatt-Smith 1988 for further details).

The consequence of this is that if we move continuously clockwise round the closed

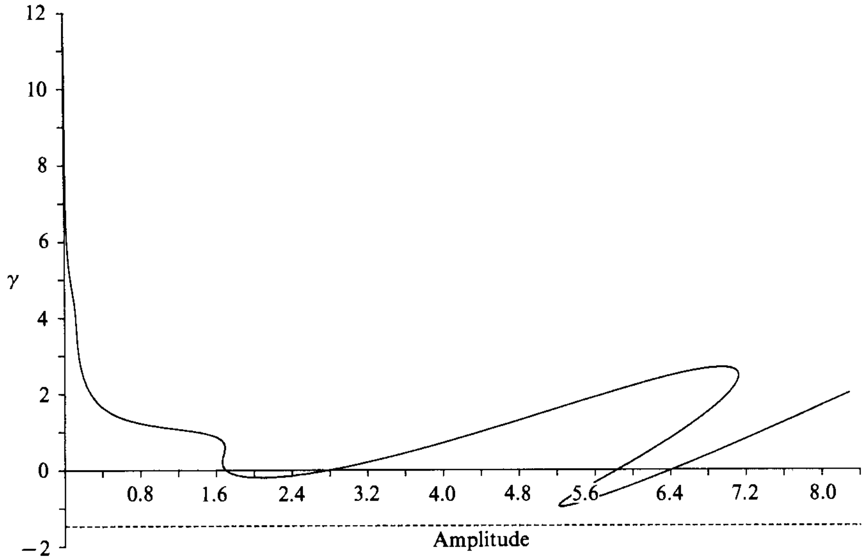

FIGURE 9. The γ–A response curve for $\epsilon = 0.1$, $\Delta = 3.5$.

loop of Γ_+ the value of ϕ_+ returns to its original value while the corresponding value of ϕ_- has decreased by 2π.

Under this process $\Delta\phi = \phi_+ + \phi_-$ decreases by a total of 2π with each complete execution of Γ_+. Alternatively, we can restrict $\Delta\phi$ to the interval $-2\pi < \Delta\phi \leqslant 0$, in which case, by (2.51) $\Delta\phi$ is unchanged with each description of Γ_+ but n decreases by 1. For each value of n, ϵ can be calculated and an (A, ϵ) response curve calculated as $\Delta\phi$ assumes all values in $(-2\pi, 0]$. In view of the continuous evolution of $\Delta\phi$ when not restricted to $(-2\pi, 0]$ the (A, ϵ) response curves for n and $n+1$ blend continuously into one another so that the assignment of a value of n to a particular portion of the curve is somewhat imprecise. For $\gamma = 1$ the critical value of $\Delta = e^{\pi\delta}$ lies between 3 and 3.5 and the curves Γ_0 for $\delta = 0$ ($\Delta = 1$) and Γ_- and Γ_+ for $\Delta = 3$ and $\Delta = 3.5$ are shown in figures 5(a), 5(b) and 5(c) respectively. The corresponding curves in the (A, ϵ) plane are plotted in figures 6(a), 6(b) and 6(c). In figures 6(a) and 6(b) the curves for different values of n are all disjoint but join at the critical value. In figure 6(c) we have one continuous family of solutions described above and also a disjoint set of curves. At either end of each of the disjoint parts of the curve the corresponding values of n effectively differ by one.

The corresponding response diagrams $\gamma(A_+)$ and $\gamma(A)$ for $\Delta = 3.5$ are shown in figures 9 and 10. The distinctive feature in comparison with figures 7 and 8 are that the resonance horns have disappeared. Whilst this is a common result of the inclusion of damping terms, some explanation is required in the light of figure 6(c). Although the drift to the right of the (A, ϕ)-curve, as explained in §3.1 still occurs, that portion of $\Gamma_-(\gamma)$ which tends to the $A = 0$ axis with increasing γ ensures a continuous small-amplitude solution as γ increases. However it is not immediately clear what has happened to the large-amplitude solution and how it can connect with the small-amplitude solution.

The first part of the explanation lies in the fact that the minimum value of A_+ on the large-amplitude portion of figure 6(c) now increases with γ. This counteracts the tendency of the drift to the right in figure 5(c), resulting in a maximum value of ϵ

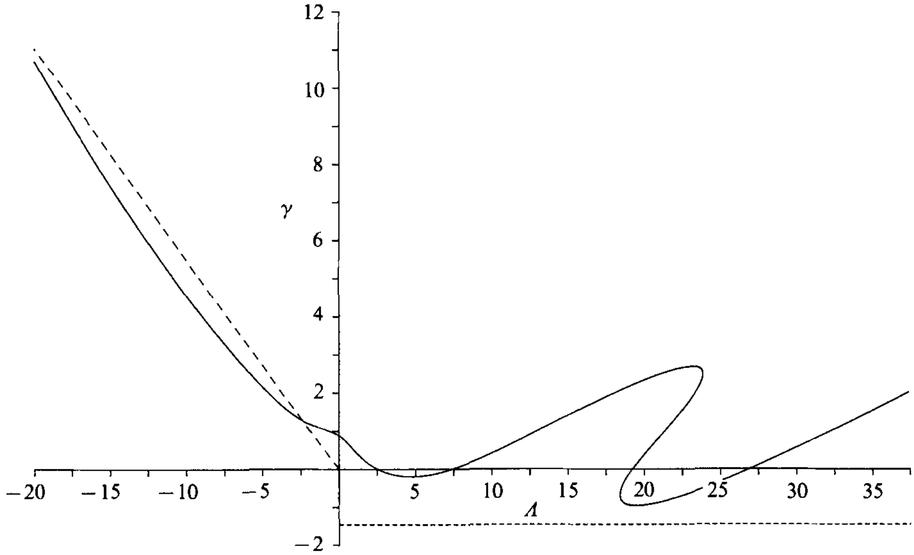

FIGURE 10. The γ–Λ response curve for $\epsilon = 0.1$, $\Delta = 3.6$.

on each curve which decreases as γ increases. The second part of the explanation is that as γ decreases there is a critical value of γ where figure 6(*c*) changes to figure 6(*b*), thus allowing the large-amplitude solution to join smoothly with the continuation of the small-amplitude solution.

4. Conclusion

We have presented an asymptotic and numerical method for constructing a solution of (2.1) which describes the periodic sloshing in a shallow rectangular tank as $\epsilon \to 0$, this limit being appropriate for the case when the water depth tends to zero. For a fixed but small value of ϵ the response diagrams $\gamma(\Lambda)$ have been calculated numerically and plotted in figures 8 and 10, representing the cases of zero and non-zero damping respectively. Here γ is given by $c\epsilon^{-\frac{4}{3}}2^{\frac{2}{3}}$ with $1+c$ being the mean-square disturbance, and the quantity Λ represents the scaled detuning parameter.

The response diagram for zero damping (figure 8) suggests that for any fixed value of Λ there is an infinite number of possible solutions which can be labelled by $(n, 0)$ or $(n, 1)$, $n \geqslant N(\Lambda)$ with n being the number of spikes corresponding to rapid oscillation on the fast time-scale. As n tends to infinity then the values of γ corresponding to the $(n, 0)$ and $(n, 1)$ solutions tend to infinity. However the amplitude of the associated oscillations remains of order $\epsilon^{\frac{5}{6}}$ corresponding to a solution of the connection problem with A of order one (see figure 7). As Λ decreases the two branches of the $(n, 0)$ and $(n, 1)$ solutions for a fixed value of n merge at $\Lambda = \Lambda_n$ to form a continuous curve. As Λ tends to infinity the values of γ corresponding to $(n, 0)$ and $(n, 1)$ also tend to infinity. This is associated with an increase of A obtained from the connection problem. This means that as Λ increases (for fixed n) the amplitude of the associated oscillations increases so that the linearized approximation of (2.19) ceases to be valid.

The corresponding response curve for the damped case, figure 10, shows that the $(n, 1)$ and the $(n+1, 0)$ meet at a finite value Λ^n rather than both branches extending to infinity. Thus in the damped case there is only a finite number of solutions for a fixed value A. Both of these response diagrams complement those obtained by Ockendon *et al.* (1986) who do not treat the case with c small.

Thus we have established the existence of a set of solutions for the periodic sloshing in a rectangular tank, which has been the purpose of this paper. In the case where multiple solutions exist we have not considered the problem of which solutions are preferred. This presumably requires knowledge of the stability of each solution and the domain of attraction in the space of initial conditions.

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Appendix. Calculation of the detuning parameter λ

From (3.7) λ is defined as

$$\lambda = \int_{-\pi}^{\pi} y \, \mathrm{d}t. \tag{A 1}$$

Since $y(\epsilon, t)$ has two different expansions we split the range of integration into two parts and define

$$\lambda_1(\rho, \epsilon) = \int_{-\pi+\rho}^{\pi-\rho} y \, \mathrm{d}t \tag{A 2}$$

and

$$\lambda_2(\rho, \epsilon) = \int_{-\pi}^{-\pi+\rho} y \, \mathrm{d}t + \int_{\pi-\rho}^{\pi} y \, \mathrm{d}t \equiv \int_{-\pi-\rho}^{-\pi+\rho} y \, \mathrm{d}t, \tag{A 3}$$

since $y(\epsilon)$ is 2π-periodic.

We assume $\rho \ll 1$ and can be chosen so that the expansion of (2.21) is valid in (A 2) and the expansion of (2.44) is valid in (A 3). The resulting expansions for λ_1 and λ_2 are not uniformly valid as ρ and ϵ tend independently to zero, but their sum, which is independent of ρ, will be. Hence a term for (A 2) that tends to infinity for fixed ϵ as $\rho \rightarrow 0$ must be cancelled by an appropriate term from (A 3).

Introducing the scaling of (2.42) into (A 3) we have

$$\lambda_2 = 2^{-\frac{1}{6}} \epsilon^{\frac{4}{3}} \int_{-R}^{R} x(T) \, \mathrm{d}T. \tag{A 4}$$

The upper limit R is given by the expression

$$R = 2^{-\frac{1}{6}} \epsilon^{-\frac{2}{3}} \rho, \tag{A 5}$$

which is assumed to be $\gg 1$ (so that $\rho \gg \epsilon^{\frac{2}{3}}$).

The integrals of all of the oscillating terms of (2.46) converge as $R \rightarrow \infty$. The only terms that produce divergent integrals are the leading term, $-T$, and the term $-\gamma/2T$ coming from $\bar{x}_0(T)$. This latter term is the one that has been seen to arise from the expansion of $(1 + \cos t + c)^{\frac{1}{2}}$.

Guided by these terms we now define

$$\begin{aligned} I(R, \epsilon) &= \int_{-R}^{R} (x_0(T) + T) \, \mathrm{d}T + \gamma \log R \\ &= \int_{-\alpha}^{\alpha} (x_0 + T) \, \mathrm{d}T + \int_{\alpha}^{R} + \int_{-R}^{-\alpha} \left(x_0 + T + \frac{\gamma}{2T} \right) \mathrm{d}T + \gamma \log \alpha. \end{aligned} \tag{A 6}$$

This integral now converges as $R \to \infty$ and we write

$$I(\epsilon) = \lim_{R\to\infty} I(R,\epsilon) = \int_{-\alpha}^{\alpha} (x_0+T)\,\mathrm{d}T + \int_{\alpha}^{\infty} + \int_{-\infty}^{-\alpha} \left(x_0+T+\frac{\gamma}{2T}\right)\mathrm{d}T + \gamma \log \alpha. \quad \text{(A 7)}$$

From (2.46) we find that

$$I(\epsilon) - I(R,\epsilon) = O(R^{-\frac{3}{4}}) \equiv O(\epsilon^{\frac{1}{2}}\rho^{-\frac{3}{4}}) \quad \text{as } R \to \infty \qquad \text{(A 8)}$$

provided $\rho \gg \epsilon^{\frac{2}{3}}$.

Then

$$\begin{aligned}\lambda_2 &= 2^{-\frac{1}{6}}\epsilon^{\frac{4}{3}}\{I(\epsilon) - R^2 - \gamma \log R + O(R^{-\frac{3}{4}})\} + o(\epsilon^{\frac{4}{3}}) \\ &= 2^{-\frac{1}{6}}\epsilon^{\frac{4}{3}}\{I(\epsilon) - 2^{-\frac{1}{3}}\epsilon^{-\frac{4}{3}}\rho^2 - \gamma \log 2^{-\frac{1}{6}}\epsilon^{-\frac{2}{3}}\rho + O(\epsilon^{\frac{1}{2}}\rho^{-\frac{3}{4}})\} + o(\epsilon^{\frac{4}{3}}) \\ &= 2^{-\frac{1}{6}}\epsilon^{\frac{4}{3}}I(\epsilon) - 2^{-\frac{1}{2}}\rho^2 - 2^{-\frac{1}{6}}\epsilon^{\frac{4}{3}}\gamma \log(2^{-\frac{1}{6}}\epsilon^{-\frac{2}{3}}\rho) + O(\epsilon^{\frac{11}{6}}\rho^{-\frac{3}{4}}) + o(\epsilon^{\frac{4}{3}}). \qquad \text{(A 9)}\end{aligned}$$

Before calculating λ_1 we first notice that the $O(\epsilon^{\frac{5}{6}})$-term in the outer solution listed in (2.31) gives rise to an integral of order $\epsilon^{\frac{11}{6}}\rho^{-\frac{3}{4}}$ which, by the remark following (A 3), is to cancel the corresponding term in (A 9). The next term in the expansion is of order $\epsilon^{\frac{4}{3}}$ and we may write

$$\begin{aligned}\lambda_1 &= \int_{-\pi+\rho}^{\pi-\rho} \left\{-\sqrt{2}\cos\tfrac{1}{2}t - \frac{2^{-\frac{2}{3}}\gamma\epsilon^{\frac{4}{3}}}{2^{\frac{3}{2}}\cos\frac{1}{2}t} + o(\epsilon^{\frac{4}{3}})\right\}\mathrm{d}t \\ &= -4\sqrt{2}\cos\tfrac{1}{2}\rho - 2^{-\frac{1}{6}}\gamma\epsilon^{\frac{4}{3}}\log\{\operatorname{cosec}\tfrac{1}{2}\rho + \cot\tfrac{1}{2}\rho\} + o(\epsilon^{\frac{4}{3}}) \\ &= -4\sqrt{2} + \tfrac{1}{2}\sqrt{2}\rho^2 + \dots + 2^{-\frac{1}{6}}\gamma\epsilon^{\frac{4}{3}}\log(\tfrac{1}{2}\rho)\,(1+\dots) + o(\epsilon^{\frac{4}{3}}). \qquad \text{(A 10)}\end{aligned}$$

Adding (A 9) and (A 10) we obtain

$$\lambda = -4\sqrt{2} - \frac{2^{-\frac{1}{6}}}{3}\gamma\epsilon^{\frac{4}{3}}\log(2^{\frac{5}{2}}\epsilon^{-2}) + 2^{-\frac{1}{6}}\epsilon^{\frac{4}{3}}I(\epsilon) + o(\epsilon^{\frac{4}{3}}). \qquad \text{(A 11)}$$

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